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A Spectral Fractional–Order Method for Solving Nonlinear Weakly Singular Volterra Integral Equations

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ABSTRACT

In this paper, we present a spectral method based on fractional–order Jacobi functions and their new constructed operational matrix of fractional integration for solving nonlinear weakly singular Volterra integral equations which include Abel equations. A proposed error analysis investigates the convergence of mentioned method. Some numerical examples are presented to confirm the efficiency and accuracy of the method.

Keywords: Operational matrix of fractional integration, weakly singular Volterra equations.

1. Introduction

Nowadays, fractional integral equations, as generalizations of ordinary integral equations, have attracted considerable attention for modeling complex phenomena in science and engineering, including continuum and statistical mechanics (Mainardi (1997), Gorenflo and Mainardi (2008)), dynamics of viscoelastic material (Meral et al. (2010)), control theory (Podlubny (2003)), electromagnetism (Engheta (1996)), fluid mechanics (Kulish and Lage (2002)) and etc.

A special case of Volterra integral equations is Abel integral equation. This type of integral equations can be derived directly from a physical problem, so it has different interdisciplinary applications in modeling environment and science phenomena, like stellar winds (Knill et al. (1993)), plasma diagnostics (Sukri et al. (2015)), nuclear physics (Kosarev (1980)), atomic scattering (Sottoni et al. (1979)), radar ranging (Muhleman et al. (1965)), microscopy (Jakeman and Anderssen (1975)), optical fiber evaluation (Liu et al. (2004)), X-ray radiography (Deutsch et al. (1990)), spectrographic data (Buie et al. (1996), Cremers and Birkebak (1966)), seismology (Jerri (1999)), metallurgy and chemical reactions (Heck and Chandler (1995)), heat conduction (Cannon (1963)) and semi-conductors (Balaban et al. (2009)).

Surveys on finding numerical schemes for solving these equations have become an active interest in a variety of fields in science. In Diogo et al. (2006), authors have proposed an approximate solution for a nonlinear Volterra integral equation of Abel type. For solving the first Abel integral equation, Lui and Tao (2007) have applied a mechanical quadrature method. Pandey et al. (2009) have introduced efficient algorithms for approximating the solution of singular integral equations of Abel type. Homotopy perturbation method has been used for solving a system of generalized Abel integral equations by Kumar et al. (2011). In Saeedi et al. (2011), authors have employed a method to solve the first and second kind Volterra integral equations with weakly singular kernel. In Shahsavaran (1996), Block–pulse functions and Taylor expansion using the collocation method has been adopted to solve the second kind of Volterra integral equations of Abel type.

Spectral and pseudo-spectral methods have been considered by many researchers due to their accurate results in scientific computations (Banerjee et al. (2019), Delkhosh and Parand (2019)). A spectral iterative method have been introduced to solve nonlinear singular Volterra integral equations of Abel type by Shoja et al. (2017). A multistep Legendre pseudo-spectral method have been proposed for solving nonlinear Volterra integral equations in Xiao-Yong and Jun-Lin (2020).

The aim of this paper is to introduce an operational fractional–order Jacobi method for solving the generalized linear and nonlinear Abel integral equations of the following forms,

$$f(t)=\int_0^t \frac{k(t,x)}{(t-x)^{\nu}} f^m(x)dx, 0<\nu<1, m\geq 1, m\in\mathbb{N}, \tag{1}$$

$$f(t)=g(t)+\int_0^t \frac{k(t,x)}{(t-x)^{\nu}} f^m(x)dx, 0<\nu<1, m\geq 1, m\in\mathbb{N}, \tag{2}$$

where these are known as the first kind and the second kind generalized Abel integral equations, respectively.

This paper is organized as follows. In order to approximate the unknown function in equations, the fractional Jacobi functions will be introduced in section 2. Section 3 is related to the extraction of the operational matrices of fractional integration. A convergence theorem will be presented in section 4. Section 5 describes the method. For demonstrating applicability and validity of the method to solve generalized Abel integral equation some examples will be provided in section 6. The last section will include the conclusions with some suggestions for further studies.

2. Fractional Calculus and Fractional–Order Jacobi Functions

In this section, we introduce the fractional integration and fractional Jacobi functions.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\eta\geq 0$ has the form (Miller and Ross (1993), Podlubny (1998), Samko et al. (1993)),*

$$I^{\eta}u(x)=\begin{cases}\frac{1}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}u(t)dt, x>0, & if \quad \eta>0,\\ u(x), & if \quad \eta=0,\end{cases}$$

where

$$\Gamma(\eta)=\int_0^{\infty} x^{\eta-1}e^{-x}dx.$$

This fractional operator has the following useful properties as follows,

$$\mathrm{I}^{\mu_1} I^{\mu_2} u(x) = I^{\mu_2} I^{\mu_1} u(x), I^{\mu_1} I^{\mu_2} u(x) = I^{\mu_1+\mu_2} u(x), I^{\eta} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\eta+1)} x^{\mu+\eta}.$$

2.1 Fractional-order Jacobi functions

Jacobi polynomials, as a family of the basis functions, which are defined on the interval $[-1, 1]$, have the following analytical form (Datta and Mohan (1995)),

$$\mathrm{p}_m^{(\alpha,\beta)}(x) = \tfrac{\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+m+1)} \sum_{k=0}^{m} \tfrac{(-1)^{k-m}\Gamma(\alpha+\beta+m+k+1)}{k!(m-k)!\Gamma(\beta+k+1)} (\tfrac{x+1}{2})^k,$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta \geq -1$. These polynomials construct an orthogonal set with respect to the weight function $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ i.e,

$$\langle p_{j_1}^{(\alpha,\beta)}(x), p_{j_2}^{(\alpha,\beta)}(x)\rangle_{w^{(\alpha,\beta)}} = \int_{-1}^{1} p_{j_1}^{(\alpha,\beta)}(x) p_{j_2}^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = h_j^{(\alpha,\beta)} \delta_{j_1 j_2},$$

$$j = j_1 = j_2,$$

where

$$h_j^{(\alpha,\beta)} = \frac{2^{(\alpha+\beta+1)}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)},$$

and $\delta_{j_1 j_2}$ is the so called Kronecker function.

By a change of variable $t = 2x - 1$, we will have the so-called shifted Jacobi functions (SHJFs). So these functions, defined on the interval $[0, 1]$, can be shown by the following analytical form,

$$\tilde{p}_m^{(\alpha,\beta)}(x) = \frac{\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+m+1)} \sum_{k=0}^{m} \frac{(-1)^{k-m}\Gamma(\alpha+\beta+m+k+1)}{k!(m-k)!\Gamma(\beta+k+1)} x^k.$$

The family of SHJFs makes an orthogonal system on the interval $[0, 1]$ with respect to the weight function

$$\tilde{w}^{(\alpha,\beta)}(x) = x^{\beta}(1-x)^{\alpha}.$$

Proof.

$$\langle \tilde{p}_{j_1}^{(\alpha,\beta)}(x), \tilde{p}_{j_2}^{(\alpha,\beta)}(x)\rangle_{\tilde{w}} = \int_0^1 \tilde{p}_{j_1}^{(\alpha,\beta)}(x)\tilde{p}_{j_2}^{(\alpha,\beta)}(x)x^{\beta}(1-x)^{\alpha}dx,$$

$$= \textstyle\int_0^1 p_{j_1}^{(\alpha,\beta)}(2x-1)p_{j_2}^{(\alpha,\beta)}(2x-1)x^{\beta}(1-x)^{\alpha}dx.$$

$$= \int_{-1}^1 p_{j_1}^{(\alpha,\beta)}(z)p_{j_2}^{(\alpha,\beta)}(z)(1-\frac{z+1}{2})^{\alpha}(\frac{z+1}{2})^{\beta}(\frac{1}{2}dz),$$

$$= \int_{-1}^1 p_{j_1}^{(\alpha,\beta)}(z)p_{j_2}^{(\alpha,\beta)}(z)(\frac{1}{2})^{\alpha+\beta+1}(1-z)^{\alpha}(1+z)^{\beta}dz,$$

$$= \tilde{h}_{j_1}^{(\alpha,\beta)}\delta_{j_1 j_2},$$

where we have used the change of variable $z = 2x - 1$, and

$$\tilde{h}_{j_1}^{(\alpha,\beta)} = (\frac{1}{2})^{\alpha+\beta+1}h_{j_1}^{(\alpha,\beta)}.$$

□

The concept of Jacobi function basis has been generalized and developed by a change of variable $t = x^{\theta}(\theta \in \mathbb{R}, 0 < \theta \leq 1)$, on SHJFs. Fractional-order Jacobi functions (FJFs), which are denoted by $p_j^{(\alpha,\beta,\theta)}(x)$, form an orthogonal basis as the following theorem,

Theorem 2.1. *The set of FJFs forms an orthogonal system on* $[0,1]$ *with respect to the weight function:*

$$w^{(\alpha,\beta,\theta)}(x) = \theta x^{(\beta+1)\theta-1}(1-x^{\theta})^{\alpha}.$$

Proof.

$$\int_0^1 p_{j_1}^{(\alpha,\beta,\theta)}(x)p_{j_2}^{(\alpha,\beta,\theta)}(x)\theta x^{(\beta+1)\theta-1}(1-x^{\theta})^{\alpha}dx$$

$$= \int_0^1 \tilde{p}_{j_1}^{(\alpha,\beta)}(x^\theta)\tilde{p}_{j_2}^{(\alpha,\beta)}(x^\theta)(x^\theta)^\beta(1-x^\theta)^\alpha \theta x^{\theta-1}dx.$$

A change of variable $x^\theta = z$ will give,

$$\int_0^1 \tilde{p}_{j_1}^{(\alpha,\beta)}(z)\tilde{p}_{j_2}^{(\alpha,\beta)}(z)z^\beta(1-z)^\alpha dz = \tilde{h}_{j_1}^{(\alpha,\beta)}\delta_{j_1 j_2}.$$

So, the result is clear. □

2.2 Function approximation

Consider the following vector space,

$$L^2_{\tilde{w}}[0,1]{:=}\{f|f \text{ is measurable on } [0,1] \text{ and } ||f||_{\tilde{w}} < \infty\},$$

equipped with the following inner product:

$$\langle f, \phi\rangle_{\tilde{w}} = \int_0^1 f(x)\phi(x)\tilde{w}(x)dx.$$

Any function $f \in L^2[0,1]$ can be expanded via FJFs as follows:

$$f(x) = \sum_{k=0}^{\infty} c_k p_k^{(\alpha,\beta,\theta)}(x),$$

where

$$c_k = \frac{1}{h_k^{(\alpha,\beta,\theta)}} \int_0^1 f(x)p_k^{(\alpha,\beta,\theta)}(x)w^{(\alpha,\beta,\theta)}(x)dx,$$

and

$$h_k^{(\alpha,\beta,\theta)} = \tilde{h}_k^{(\alpha,\beta)}. \tag{3}$$

Using the truncated series, we have,

$$f(x) \simeq \sum_{k=0}^{M-1} c_k p_k^{(\alpha,\beta,\theta)}(x) = C^T\Phi_\theta^{(\alpha,\beta)}(x), \tag{4}$$

where C and $\Phi_\theta^{(\alpha,\beta)}(x)$ are the following $M-$vectors,

$$C = [c_0, c_1, c_2, \ ..., c_{M-1}]^T,$$

$$\Phi_\theta^{(\alpha,\beta)}(x) = [p_0^{(\alpha,\beta,\theta)}(x), p_1^{(\alpha,\beta,\theta)}(x), p_2^{(\alpha,\beta,\theta)}(x), \ ..., p_{M-1}^{(\alpha,\beta,\theta)}(x)]^T.$$

3. FJFs–Operational Matrices of Integration

In this section, we want to derive the operational fractional integration and product matrices related to FJFs, respectively.

Lemma 3.1. *The Riemann-Liouville fractional integral of order ν of the fractional–order Jacobi functions vector $\Phi_\theta^{(\alpha,\beta)}(t)$ is obtained by,*

$$I^\nu \Phi_\theta^{(\alpha,\beta)}(t) \approx Q^\nu \Phi_\theta^{(\alpha,\beta)}(t),$$

where Q^ν is the fractional integral operational matrix of order ν and for $s \neq j$, $q(i,j) = 0$, where for $s = j$, the elements of the matrix can be calculated as the following,

$$\begin{aligned} q(i,j) = \tfrac{\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+1)} \sum_{k=0}^{i} \frac{(-1)^{k-i}\Gamma(\alpha+\beta+i+k+1)}{k!(i-k)!\Gamma(\beta+k+1)} \frac{\Gamma(k\theta+1)}{\Gamma(k\theta+\nu+1)} \\ \times \sum_{s=0}^{M-1} \frac{1}{h_s^{(\alpha,\beta,\theta)}} \frac{\Gamma(\beta+s+1)}{\Gamma(\alpha+\beta+s+1)} \\ \times \sum_{r=0}^{s} \frac{(-1)^{r-s}\Gamma(\alpha+\beta+s+r+1)}{r!(s-r)!\Gamma(\beta+r+1)} \frac{\Gamma(\frac{\nu}{\theta}+k+r+\beta+1)\Gamma(\alpha+1)}{\Gamma(\frac{\nu}{\theta}+k+r+\beta+\alpha+2)}. \end{aligned}$$

Proof. The Rieman-Liouville integration operator can be approximated as follows,

$$\tfrac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} p_i^{(\alpha,\beta,\theta)}(x)dx = \tfrac{\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+1)} \sum_{k=0}^{i} \frac{(-1)^{k-i}\Gamma(\alpha+\beta+i+k+1)}{k!(i-k)!\Gamma(\beta+k+1)} I^\nu t^{k\theta},$$

$$= \tfrac{\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+1)} \sum_{k=0}^{i} \frac{(-1)^{k-i}\Gamma(\alpha+\beta+i+k+1)}{k!(i-k)!\Gamma(\beta+k+1)} \frac{\Gamma(k\theta+1)}{\Gamma(k\theta+\nu+1)} t^{k\theta+\nu}.$$

Now, we approximate $t^{\nu+k\theta}$ with the first M terms of fractional Jacobi functions as the following,

$$t^{\nu+k\theta} \approx \sum_{s=0}^{M-1} c_s p_s^{(\alpha,\beta,\theta)}(t),$$

where

$$c_s = \frac{1}{h_s^{(\alpha,\beta,\theta)}} \int_0^1 t^{\nu+k\theta} p_s^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t)dt.$$

Then

$$c_s = \frac{1}{h_s^{(\alpha,\beta,\theta)}} \frac{\Gamma(\beta+s+1)}{\Gamma(\alpha+\beta+s+1)} \sum_{r=0}^{s} \frac{(-1)^{r-s}\Gamma(\alpha+\beta+s+r+1)}{r!(s-r)!\Gamma(\beta+r+1)} \times \int_0^1 t^{\nu+k\theta} t^{r\theta} \theta t^{(\beta+1)\theta-1}(1-t^\theta)^\alpha dt. \quad (5)$$

Now, by a change of variable $t^\theta = z$, the expression (5) can be written as,

$$c_s = \frac{1}{h_s^{(\alpha,\beta,\theta)}} \frac{\Gamma(\beta+s+1)}{\Gamma(\alpha+\beta+s+1)} \sum_{r=0}^{s} \frac{(-1)^{r-s}\Gamma(\alpha+\beta+s+r+1)}{r!(s-r)!\Gamma(\beta+r+1)} \int_0^1 z^{\frac{\nu}{\theta}+k+r+\beta}(1-z)^\alpha dz,$$

that is

$$c_s = \frac{1}{h_s^{(\alpha,\beta,\theta)}} \frac{\Gamma(\beta+s+1)}{\Gamma(\alpha+\beta+s+1)} \sum_{r=0}^{s} \frac{(-1)^{r-s}\Gamma(\alpha+\beta+s+r+1)}{r!(s-r)!\Gamma(\beta+r+1)} B(\frac{\nu}{\theta}+k+r+\beta+1, \alpha+1),$$

where $B(x,y)$ is the so-called beta function.

Considering the relation between the gamma and beta functions, we get,

$$c_s = \frac{1}{h_s^{(\alpha,\beta,\theta)}} \frac{\Gamma(\beta+s+1)}{\Gamma(\alpha+\beta+s+1)} \sum_{r=0}^{s} \frac{(-1)^{r-s}\Gamma(\alpha+\beta+s+r+1)}{r!(s-r)!\Gamma(\beta+r+1)} \times \frac{\Gamma(\frac{\nu}{\theta}+k+r+\beta+1)\Gamma(\alpha+1)}{\Gamma(\frac{\nu}{\theta}+k+r+\beta+\alpha+2)}. \quad (6)$$

Now, $I^\nu p_i^{(\alpha,\beta,\theta)}(t)$ can be expressed in terms of FJFs basis as follows:

$$I^\nu p_i^{(\alpha,\beta,\theta)}(t) = \sum_{j=0}^{M-1} q(i,j) p_j^{(\alpha,\beta,\theta)}(t),$$

and

$$q(i,j) = \frac{\langle I^\nu p_i^{(\alpha,\beta,\theta)}(t), p_j^{(\alpha,\beta,\theta)}(t)\rangle_{w^{(\alpha,\beta,\theta)}(t)}}{\langle p_j^{(\alpha,\beta,\theta)}(t), p_j^{(\alpha,\beta,\theta)}(t)\rangle_{w^{(\alpha,\beta,\theta)}(t)}},$$

where $\langle , \rangle$ denotes the inner product in $L^2[0,1]$. Therefore,

$$q(i,j) = \frac{1}{h_j^{(\alpha,\beta,\theta)}} \frac{\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+1)} \sum_{k=0}^{i} \frac{(-1)^{k-i}\Gamma(\alpha+\beta+i+k+1)}{k!(i-k)!\Gamma(\beta+k+1)} \frac{\Gamma(k\theta+1)}{\Gamma(k\theta+\nu+1)}$$
$$\times \sum_{s=0}^{M-1} c_s \int_0^1 p_s^{(\alpha,\beta,\theta)}(t) p_j^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t)dt.$$

Considering (6) and using the orthogonal property of FJFs, the result be derived. □

3.1 Product operational matrix of two FJFs

The product of $\Phi_\theta^{(\alpha,\beta)}(t)$, $\Phi_\theta^{(\alpha,\beta)T}(t)$ and M–vector C can be stated by the new basis,

$$\Phi_\theta^{(\alpha,\beta)}(t)\Phi_\theta^{(\alpha,\beta)T}(t)C \simeq U(C)\Phi_\theta^{(\alpha,\beta)}(t), \tag{7}$$

where

$$U(C) = \langle \Phi_\theta^{(\alpha,\beta)}(t)\Phi_\theta^{(\alpha,\beta)T}(t)C, \Phi_\theta^{(\alpha,\beta)}(t)\rangle_{w^{(\alpha,\beta,\theta)}(t)}.$$

The elements of the $(M \times M)$–matrix $U(C)$, can be calculated as follows:

$$u_{i,j} = \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 p_i^{(\alpha,\beta,\theta)}(t) p_\ell^{(\alpha,\beta,\theta)}(t) c_\ell p_j^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t)\, dt, i = 0,1,\ldots,M-1,$$

$j = 0,1,\ldots,M-1, \ell = 0,1,\ldots,M-1,$

c_ℓ is the ℓ -th entry of vector C.

4. Convergence Analysis

In this section, we will state the corresponding convergence theorem.

Theorem 4.1. *Suppose that $f \in L^2[0,1]$, and $\Phi_\theta^{(\alpha,\beta)}(t)$ is a FJFs–vector. A sequence $f_{\hat{n}}(t)$ defined by $f_{\hat{n}}(t) = \sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t)$, $\hat{n} \in \mathbb{N}$, with:*

$$c_j = < f(t), p_j^{(\alpha,\beta,\theta)}(t) >_{w^{(\alpha,\beta,\theta)}}, j = 1,2,\ \ldots,\hat{n},$$

converges to $f(t)$ from the above in the vector space of $\Phi_\theta^{(\alpha,\beta)}(t)$'s components if and only if $\sum_{j=1}^{\infty} |h_j^{(\alpha,\beta,\theta)} c_j^2| < \infty$.

Proof. Let $f_{\hat{n}}$ be converges to f. Hence, for $\hat{n} \in \mathbb{N}$, we have,

$$0 \leq ||f - \sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}||_2^2$$

$$= \int_0^1 w^{(\alpha,\beta,\theta)}(t)(f(t) - \sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t))^2 \, dt$$

$$= \int_0^1 w^{(\alpha,\beta,\theta)}(t) f^2(t) dt - \int_0^1 w^{(\alpha,\beta,\theta)}(t) 2 f(t) \sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t) \, dt$$

$$+ \int_0^1 w^{(\alpha,\beta,\theta)}(t)(\sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t))^2 dt,$$

$$\leq \int_0^1 w^{(\alpha,\beta,\theta)}(t) f^2(t) dt - \int_0^1 w^{(\alpha,\beta,\theta)}(t) 2 f_{\hat{n}}(t) \sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t) \, dt$$

$$+ \int_0^1 w^{(\alpha,\beta,\theta)}(t)(\sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t))^2 dt,$$

$$= \int_0^1 w^{(\alpha,\beta,\theta)}(t) f^2(t) dt - \int_0^1 w^{(\alpha,\beta,\theta)}(t)(\sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}(t))^2 dt,$$

$$= \int_0^1 w^{(\alpha,\beta,\theta)}(t) f^2(t) dt - \int_0^1 w^{(\alpha,\beta,\theta)}(t)(c_1^2 (p_1^{(\alpha,\beta,\theta)}(t))^2 + c_2^2 (p_2^{(\alpha,\beta,\theta)}(t))^2$$

$$+ \ldots + c_{\hat{n}}^2 (p_{\hat{n}}^{(\alpha,\beta,\theta)}(t))^2) \, dt,$$

$$= ||\mathrm{f}||_2^2 - \sum_{j=1}^{\hat{n}} |h_j^{(\alpha,\beta,\theta)} c_j^2|.$$

So

$$0 \leq ||f||_2^2 - \sum_{j=1}^{\hat{n}} |h_j^{(\alpha,\beta,\theta)} c_j^2|,$$

and, for any $f \in L^2[0,1]$, we have,

$$\sum_{j=1}^{\hat{n}} |h_j^{(\alpha,\beta,\theta)} c_j^2| \leq ||f(t)||_2^2 < \infty.$$

Hence

$$\sum_{j=1}^{\infty} |h_j^{(\alpha,\beta,\theta)} c_j^2| < \infty.$$

Now, let $\sum_{j=1}^{\infty} |h_j^{(\alpha,\beta,\theta)} c_j^2| < \infty$. Then if $\hat{n} \in \mathbb{N}$, we have,

$$0 \leq ||f - f_{\hat{n}}||_2^2 = ||\sum_{j=1}^{\infty} c_j p_j^{(\alpha,\beta,\theta)} - \sum_{j=1}^{\hat{n}} c_j p_j^{(\alpha,\beta,\theta)}||_2^2,$$

$$=||\sum_{j=\hat{n}+1}^{\infty} c_j p_j^{(\alpha,\beta,\theta)}||_2^2,$$

$$=\int_0^1 w^{(\alpha,\beta,\theta)}(t)(\sum_{j=\hat{n}+1}^{\infty} c_j p_j^{(\alpha,\beta,\theta)}(t))^2\, dt,$$

$$\leq \sum_{j=\hat{n}+1}^{\infty} c_j^2 \int_0^1 w^{(\alpha,\beta,\theta)}(t)(p_j^{(\alpha,\beta,\theta)}(t))^2 dt,$$

$$= \sum_{j=\hat{n}+1}^{\infty} |h_j^{(\alpha,\beta,\theta)} c_j^2|.$$

So, as $\hat{n}$ goes to infinity, the last term will go to zero, and the proof is completed. □

5. Implementation Method

Consider (4), let $\Omega = [0, 1]$, and w is a weight function on Ω in the usual sense. Define,

$$L^2_{\tilde{w}}(\Omega){:=}\{f|f \text{ is measurable on } [0,1] \text{ and } ||f||_{\tilde{w}} < \infty\},$$

$$L^2_{\tilde{\omega}(t,x)}(\Omega \times \Omega){:=}\{f(t,x)|f \text{ is measurable on } [0,1]\times[0,1] \text{ and } ||f||_{\tilde{\omega}(t,x)} < \infty\},$$

equipped with the following inner products and norms respectively,

$$\langle f, \phi\rangle_{\tilde{w}} = \int_0^1 f(x)\phi(x)\tilde{w}(x)dx.$$

$$\langle \phi_i, \langle f(t,x), \phi_j \rangle \rangle_{\tilde{\omega}(t,x)} = \int_0^1 \int_0^1 \phi(t) f(t,x) \phi(x) \tilde{w}(t) \tilde{w}(x) dt dx.$$

Suppose $f, g \in L^2[0,1]$ and $k(t,x) \in L^2([0,1] \times [0,1])$. So we may write,

$$f(x) \approx C^T \Phi_\theta^{(\alpha,\beta)}(x), g(x) \approx G^T \Phi_\theta^{(\alpha,\beta)}(x),$$

$$k(t,x) \approx \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} k_{ij} p_i^{(\alpha,\beta,\theta)}(x) p_j^{(\alpha,\beta,\theta)}(t) = \Phi_\theta^{(\alpha,\beta)T}(t) K \Phi_\theta^{(\alpha,\beta)}(x),$$

where

$$k_{ij} = \frac{1}{h_i^{(\alpha,\beta,\theta)} h_j^{(\alpha,\beta,\theta)}} \int_0^1 \int_0^1 p_i^{(\alpha,\beta,\theta)}(t) k(t,x) p_j^{(\alpha,\beta,\theta)}(x) w^{(\alpha,\beta,\theta)}(x) w^{(\alpha,\beta,\theta)}(t) dx dt.$$

Now we present some theorems to approximate the integral part of the equations (1) and (2) for linear case ($m = 1$) and nonlinear cases ($m > 1$). theorem Suppose that $f \in C[0,1]$ and $0 < \nu < 1$; then, the integral part of equations (1) and (2) for the case $m = 1$, can be expressed in terms of FJFs–basis as follows,

$$\int_0^t \frac{k(t,x)}{(t-x)^\nu} f(x) dx \approx \sum_{j=0}^{M-1} F_j p_j^{(\alpha,\beta,\theta)}(t) = F^T \Phi_\theta^{(\alpha,\beta)}(t),$$

where

$$F_j \approx \Gamma(1-\nu) \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 \Phi_\theta^{(\alpha,\beta)T}(t) K U(C) Q^{1-\nu} p_j^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t) dt.$$

$j = 0, 1, \ldots, M-1.$

Proof.

$$F_j = \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 \int_0^t \frac{k(t,x)}{(t-x)^\nu} f(x) p_j^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t) dx\, dt.$$

So

$$F_j \approx \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 \int_0^t (t-x)^{(1-\nu)-1} (\Phi_\theta^{(\alpha,\beta)}(t)^T K \Phi_\theta^{(\alpha,\beta)}(x))(C^T \Phi_\theta^{(\alpha,\beta)}(x))^T$$

$$\times p_j^{(\alpha,\beta,\theta)}(t)w^{(\alpha,\beta,\theta)}(t)dt\,dx.$$

Using the relation (7), we have,

$$F_j \approx \frac{1}{h_j^{(\alpha,\beta,\theta)}}\int_0^1\int_0^t (t-x)^{(1-\nu)-1}(\Phi_\theta^{(\alpha,\beta)T}(t)KU(C)\Phi_\theta^{(\alpha,\beta)}(x))$$
$$\times p_j^{(\alpha,\beta,\theta)}(t)w^{(\alpha,\beta,\theta)}(t)dt\,dx.$$

We know that

$$\int_0^t (t-x)^{(1-\nu)-1}p_j^{(\alpha,\beta,\theta)}(x)dx = \Gamma(1-\nu)Q^{1-\nu}p_j^{(\alpha,\beta,\theta)}(t).$$

Therefore

$$F_j \approx \frac{1}{h_j^{(\alpha,\beta,\theta)}}\Gamma(1-\nu)\int_0^1 \Phi_\theta^{(\alpha,\beta)T}(t)KU(C)Q^{1-\nu}p_j^{(\alpha,\beta,\theta)}(t)w^{(\alpha,\beta,\theta)}(t)dt.$$

□

Lemma 5.1. *If $f(t) \simeq C^T\Phi_\theta^{\alpha,\beta}(t)$ then,*

$$f^m(t) \approx C^TU^{m-1}(C)\Phi_\theta^{(\alpha,\beta)}(t), m \in \mathbb{N},$$

where U is the product operational matrix of two FJFs.

Proof. For $m = 2$, we have,

$$f^2(t) = f(t)f(t) \approx (C^T\Phi_\theta^{(\alpha,\beta)}(t))(\Phi_\theta^{(\alpha,\beta)T}(t)C) = C^TU(C)\Phi_\theta^{(\alpha,\beta)}(t).$$

By using induction on m, the proof is completed. □

Theorem 5.1. *Suppose that $f \in C[0,1]$ and $0 < \nu < 1$, then the integral part of equations (1) and (2) for $m > 1$ can be expanded in terms of FJFs–vector as the following,*

$$\int_0^t \frac{k(t,x)f^m(x)}{(t-x)^\nu}dt \approx \sum_{j=0}^{M-1} E_jp_j^{(\alpha,\beta,\theta)}(t) = E^T\Phi_\theta^{(\alpha,\beta)}(t),$$

$$E_j \approx \frac{1}{h_j^{(\alpha,\beta,\theta)}}\int_0^1 (t-x)^{(1-\nu)-1}(\Phi_\theta^{(\alpha,\beta)T}(t)A\Phi_\theta^{(\alpha,\beta)}(t))p_j^{(\alpha,\beta,\theta)}(t)w^{(\alpha,\beta,\theta)}(t)dtdx,$$

where $A = KU(C_1)\Gamma(1-\nu)Q^{1-\nu}$, $C_1^T = C^TU^{m-1}(C)$, and $j = 0, 1, \ldots, M-1$.

Proof. We know,

$$E_j = \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 \int_0^t \frac{k(t,x)}{(t-x)^{\nu}} f^m(x) p_j^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t) dt dx,$$

So

$$E_j \approx \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 \int_0^t (t-x)^{(1-\nu)-1} (\Phi_{\theta}^{(\alpha,\beta)T}(t) K \Phi_{\theta}^{(\alpha,\beta)}(x))(C^T U^{m-1}(C) \Phi_{\theta}^{(\alpha,\beta)}(x))$$

$$\times p_j^{(\alpha,\beta,\theta)}(t) w^{(\alpha,\beta,\theta)}(t) dt\, dx.$$

Let $C^T U^{m-1}(C) = C_1^T$, so by using (5.2), we can write,

$$E_j \approx \frac{1}{h_j^{(\alpha,\beta,\theta)}} \int_0^1 \int_0^t (t-x)^{(1-\nu)-1} (\Phi_{\theta}^{(\alpha,\beta)T}(t) K U(C_1) \Gamma(1-\nu) Q^{1-\nu} \Phi_{\theta}^{(\alpha,\beta)}(x)) p_j^{(\alpha,\beta,\theta)}(t)$$

$$\times w^{(\alpha,\beta,\theta)}(t) dt\, dx.$$

□

Now, equation (2), for $m > 1$ can be rewritten as follows,

$$C^T \Phi_{\theta}^{(\alpha,\beta)}(t) \simeq G^T \Phi_{\theta}^{(\alpha,\beta)}(t) + E^T \Phi_{\theta}^{(\alpha,\beta)}(t).$$

By using the orthogonal property of FJFs, we have the following system of nonlinear equations:

$$C = G + E, \tag{8}$$

which can be solved using Newton-Raphson method. By substituting the derived vector C in (8), the solution of (2) can be obtained.

6. Numerical Results

In this section, some examples will be presented to show the efficiency and applicability of the method. Numerical examples are considered in both linear and nonlinear cases. Let $f(t)$ and $f_n(t)$ be the exact and approximate solutions of the main equation, respectively. The error function is defined as $e_n(t) = f(t) - f_n(t)$.

Example 1. Consider the following weakly singular Volterra integral equation,

$$f(t) = 2\sqrt{t} - \int_0^t \frac{f(x)}{\sqrt{t-x}} dx, 0 \le t \le 1, \tag{9}$$

in which the exact solution is,

$$f(t) = 1 - e^{\pi t} erfc(\sqrt{\pi t}),$$

where

$$erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2} dx.$$

The solution of this equation is approximated by applying the present method. Figure 1 shows error function in different cases of α and β and $M = 9$.

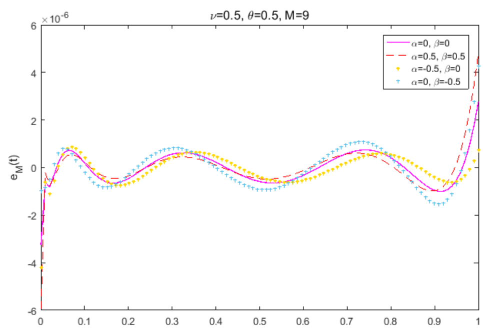
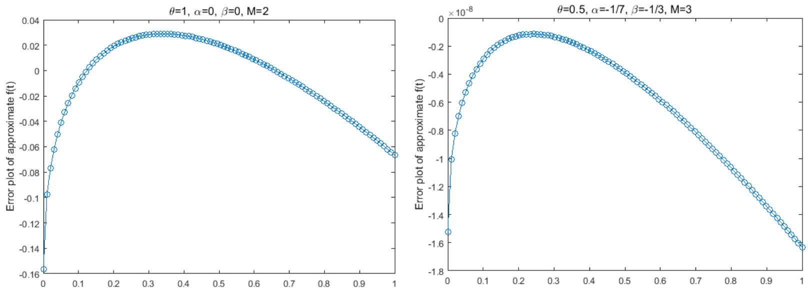


Figure 1: The error of approximate solutions for different cases of (α, β) (Example 1)

Example 2. Consider the following first kind Abel integral equation.

$$\int_0^t \frac{f(x)}{\sqrt{t-x}} dx = \pi, 0 \le t \le 1, \tag{10}$$

in which the exact solution is $f(t) = \frac{1}{\sqrt{t}}$.



(a) (b)

Figure 2: The error of approximate solutions for different cases of (α, β), and M (Example 2)

The solution of this equation is approximated by applying our method. Figure 2 shows error function in different cases of α and β and M.

Example 3. Consider the following Abel integral equation,

$$f(t) = g(t) - \int_0^t \frac{f(x)}{\sqrt{t-x}} dx, 0 \leq t \leq 1, \tag{11}$$

$$g(t) = cos(t) + sin(t) + \sqrt{2\pi}(fresnels(\sqrt{\frac{2t}{\pi}}(-cos(t) + sin(t)) +$$
$$sin(t)) + fresnelc(\sqrt{\frac{2t}{\pi}}(cos(t) + sin(t)))).$$

where

$$fresnels(u) = \int_0^u sin(\tfrac{\pi s^2}{2})ds, \text{ and, } fresnelc(u) = \int_0^u cos(\tfrac{\pi s^2}{2})ds,$$

and the exact solution is $f(t) = sin(t) + cos(t)$.

The solution of this equation is approximated by applying our method. Figure 3 shows the error function in different cases of M, $\alpha = \beta = 0$ and $\theta = 0.5$. Furthermore, we know that Jacobi functions by changing the values of α and β include a wide clases of bases, including Legendre base functions $(\alpha = \beta = 0)$, Chebyshev base functions of the first kind $(\alpha = \beta = -0.5)$, Chebyshev base functions of the second kind $(\alpha = \beta = 0.5)$, and ultraspherical functions as Gegenbauer base functions $(\alpha = \beta)$. Table 1 compares the error of the approximate solution of Example 3 with $\nu = 0.5$, and $\nu = 0.7$, for Legendre basis and the second kind of Chebyshev basis respectively.

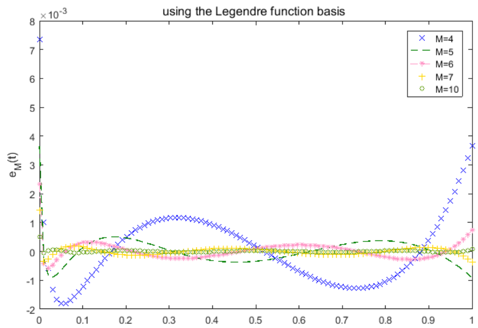


Figure 3: The error of approximate solutions for different values of M.

Table 1: Comparing the errors of approximate solutions for different cases of (α, β), for $I^{0.5}$ and $I^{0.7}$ (Example 3)

	Legendre basis	*Chebyshev basis*	*Legendre basis*	*Chebyshev basis*
x=0.1	-1.922e-08	-8.419e-08	-3.6806e-05	-2.309e-05
x=0.2	1.238e-08	-4.430e-08	2.907e-05	1.624e-05
x=0.3	1.709e-08	-2.843e-08	-1.934e-05	-9.661e-05
x=0.4	-3.165e-08	-5.518e-08	-6.656e-06	-3.446e-06
x=0.5	-5.492e-09	-2.350e-08	2.257e-05	1.225e-05
x=0.6	3.796e-08	9.661e-09	-1.830e-06	-5.535e-06
x=0.7	-2.869e-09	-2.988e-08	-1.922e-05	-1.070e-05
x=0.8	-4.548e-08	-4.354e-08	2.168e-05	1.585e-05
x=0.9	5.324e-08	5.884e-08	-1.312e-05	-1.290e-05

Example 4. Consider the following nonlinear weakly singular Volterra integral equation:

$$f(t) + \int_0^t \frac{f^5(x)}{\sqrt{t-x}} dx = g(t), 0 \leq t \leq 1, \tag{12}$$

where $f(t) = t$ is the exact solution and $g(t) = t + \frac{512}{693} t^{\frac{11}{2}}$.

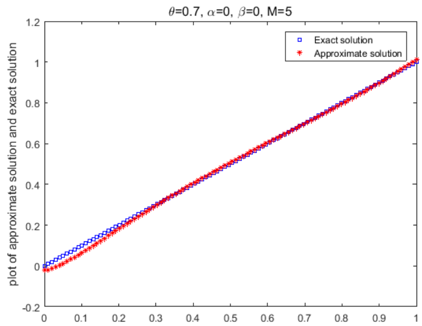


Figure 4: Approximate and exact solutions for Example 4 with $\theta = 0.7$ and $M = 5$, (Example 4)

The numerical results are shown in Figure 4.

Example 5. Consider the following Volterra integral equation with nonlinear weakly singular kernel:

$$f(t) + \int_0^t \frac{k(x,t)f^2(x)}{(t-x)^{\frac{1}{3}}}dx = g(t) \tag{13}$$

with $f(t) = \sqrt{t}$, as the exact solution, and $k(x,t) = xt$, and also $g(t) = \sqrt{t} + \frac{27}{40}t^{\frac{11}{3}}$.

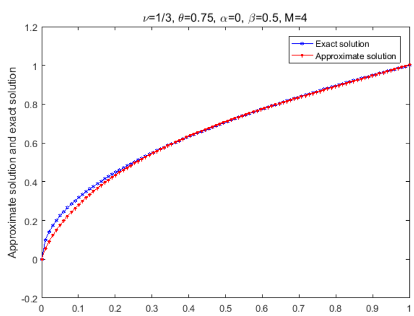


Figure 5: Approximate and exact solutions for Example 5

The numerical results are portrayed in Figure 5.

7. Conclusions

In this paper, the authors employed a spectral method for solving generalized linear and nonlinear weakly singular Volterra integral equation of Abel type. Our operational method, based on the fractional–order Jacobi functions, converts the problem to a system of linear and nonlinear equations. Finally, the numerical examples have been provided to guarantee the applicability and accuracy of the method. This method can be used for other integral equations as well.

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